of the critical force.

These deductions have been obtained independently of the properties of the plate material, and refer equally to both isotropic, and transversely isotropic materials with small shear stiffness, and to both nonlinearly elastic and elastoplastic materials if the concept of a tangent modulus is assumed in the latter cases.

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STATISTICAL DETERMINATION OF THE TENSOR OF VISCOSITY COEFFICIENTS

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An expression for the tensor of viscosity coefficients in the terms of the autocorrelation functions is obtained under the assumptions of the Kubo theory [1] of the linear reaction of a system subjected to a mechanical perturbation. These coefficients are obtained as components of a fourth rank tensor for an arbitrary homogeneous anisotropic medium using the framework of the Gibbs formalism without, however, employing the well-known additional representations. Coefficient of the shear viscosity of an isotropic medium is determined to illustrate the proposed method of computing the integrals of autocorrelation functions. This method uses the concept of statistical averaging in the state of equilibrium and utilizes mean relaxation times. Double index correlative distribution functions are used to obtain, by statistical methods, the relaxation times for the impulse-dependent quantities and for the spatial coordinate-dependent magnitudes. Numerical estimates for simple fluids show, that the impulse relaxation time is of the order of 10^{-14} sec, while the coordinate relaxation time is of the order of 10^{-12} sec.

Let an isolated system be acted upon by a small mechanical perturbation causing its Hamiltonian H to change by the amount $\Delta H = -AF(t)$. Then by the Kubo theory, the mean value $\Delta B(t)$ of the variation of the dynamic variable $B(t)$ is given by

$$\Delta B(t) = -\frac{1}{\theta}\int_0^\infty < A(0)\, B^{\cdot}(s)\rangle\, F(t-s)\, ds \qquad \left(\theta = kT,\ B^{\cdot} = \frac{dB}{dt}\right) \tag{1}$$

where the symbol $\langle\rangle$ denotes averaging over the stable canonical ensemble.

To investigate the viscous properties of the medium, we shall study its behavior under a small strain

$$x_n{}^+ = x_n + \frac{\partial u_n}{\partial q^m} x_m \tag{2}$$

Here x_n denotes a component of the radius vector $\mathbf{r} = \mathbf{q}^2 - \mathbf{q}^1$ connecting two particles prior to deformation and $x_n{}^+$ – after the deformation; u_n is a component of the displacement vector of the particle and the impulses of the particles transform according to the Lagrangian formalism

$$p_n = p_m{}^+ \frac{\partial x_m{}^+}{\partial x_n} \tag{3}$$

Summation is performed over like indices.

Solving the system (3) for the impulse components of the particle after the deformation, we can write

$$p_i{}^+ = p_i - \frac{\partial u_i}{\partial q^k} p_k + \cdots \tag{4}$$

where the dots denote the second and higher order terms $\partial u_i / \partial q^k$.

For a system of N particles interacting pairwise and described by the intermolecular potential $\Phi(r)$, the Hamiltonian has the form

$$H_N = \sum_{\nu=1}^{N} \frac{1}{2m}(p_i{}^\nu)^2 + \frac{1}{2}\sum_{\nu,\mu} \Phi(r^{\mu\nu}) \tag{5}$$

where m is the mass of a particle; ν and μ denote the νth and μth particle.

Then $\Delta H = H_N{}^+ - H_N$ is given, with the second and higher order terms neglected, by

$$\Delta H = \Pi^+_{mn} \frac{\partial u_n}{\partial q^m} \tag{6}$$

where

$$\Pi^+_{mn} = -\sum_{\nu=1}^{N} \frac{p_n{}^\nu p_m{}^\nu}{m} + \frac{1}{2}\sum_{\nu,\mu} \frac{\Phi'(r)\, x_m{}^{\mu\nu} x_n{}^{\mu\nu}}{r} \tag{7}$$

Taking into account the symmetry of the tensor $\Pi_{mn}{}^+$ we can obtain ΔH in the form

$$\Delta H = \Pi^+_{mn} u_{mn}, \qquad u_{mn} = \frac{1}{2}\left(\frac{\partial u_n}{\partial q^m} + \frac{\partial u_m}{\partial q^n}\right) \tag{8}$$

Thus, $A = -\Pi_{mn}{}^+$ and $F = u_{mn}$. Putting

$$B = \int_{-\infty}^{t} \Pi_{ik}{}^+(t)\, dt \tag{9}$$

we obtain, by (1),

$$\Delta B = \frac{1}{\theta}\int_0^\infty \langle \Pi_{ik}{}^+(0)\, \Pi^+_{mn}(s)\rangle\, u_{mn}(t-s)\, ds \tag{10}$$

Let us now suppose that the deformation is cyclic

$$u_{mn}(t) = u_{mn}(0)\, e^{i\omega t} \tag{11}$$

Then

$$\Delta B = \left[\frac{1}{\theta}\int_0^\infty e^{-i\omega s} \langle \Pi_{ik}{}^+(0)\, \Pi^+_{mn}(s)\rangle ds\right] u_{mn}(t) \tag{12}$$

Let us now compare (12) with the following well known phenomenological relation

$$\int_{-\infty}^{t} \sigma_{ik}\, ds = \eta_{ikmn} u_{mn} \tag{13}$$

where $\sigma_{ik} = \eta_{ikmn}\, u_{mn}$, σ_{ik} being the viscous stress tensor which is assumed to tend to zero as $t \to -\infty$. This yields the following expression for the tensor of viscosity coefficients:

$$\eta_{ikmn} = \frac{1}{\theta} \int_{0}^{\infty} e^{-i\omega s} \langle \Pi_{ik}^{+}(0)\, \Pi_{mn}^{+}(s) \rangle\, ds \tag{14}$$

It should be noted that the flux $\Pi_{ik}{}^{+}$ is defined with the accuracy of up to any quantity of zero divergence.

To remove this lack of definiteness, we can rewrite [2] Expression (14) as follows (the canonical ensemble is used for averaging):

$$\eta_{ikmn} = \frac{1}{\theta} \int_{0}^{\infty} e^{-i\omega s} \langle (\Pi_{ik}^{+}(0) - \overline{\Pi}_{ik}(0))\, (\Pi_{mn}^{+}(s) - \overline{\Pi}_{mn}(s)) \rangle\, ds \tag{15}$$

$$\overline{\Pi}_{ik} = \sigma_{ik}{}^{\circ} + \frac{\partial \sigma_{ik}{}^{\circ}}{\partial \overline{E}} (H_N - \overline{E})$$

Here $\sigma_{ik}{}^{\circ}$ denotes the mean equilibrium stress tensor and $\overline{E}$ is the mean value of the total energy H_N of the system.

The above expressions agree with the results obtained in [3 and 4] where the concept of local Gibbs' distribution was used.

For an isotropic fluid, the general expression (15) yields the already known result for shear viscosity (in [5] it is based on the Feinman's concept of constructing a viscous flow by altering the dimensions of the vessel containing the set of particles)

$$\eta(\omega) = \frac{1}{\theta} \int_{0}^{\infty} e^{-i\omega t} \langle \Pi_{12}{}^{+}(0)\, \Pi_{12}{}^{+}(t) \rangle\, dt \tag{16}$$

Expression (15) can also yield a relation for the volume viscosity (the volume viscosity was obtained in [6] using the Feinman's concept). The case of low energy loading corresponds to the condition $\omega = 0$.

The familiar difficulties encountered in computing the integrals of the autocorrelation functions can be overcome by the use of approximate methods based on the concept of the mean relaxation times for the dynamic quantities appearing in the autocorrelation functions. We note that the relaxation times for the impulses and coordinates are, generally speaking, different. Therefore, separate relaxation times must be introduced for the impulse (τ_p)-dependent and the coordinate (τ_q)-dependent quantities.

Let us use this point of view to consider the computation of the integral in (16). The dynamic quantity $\Pi_{12}{}^{+}$ consists, according to (7), of two terms, the impulse function $C(p)$ and the coordinate function $D(q)$. Then the integral with respect to time of $\langle \Pi_{12}{}^{+}(0) \Pi_{12}{}^{+}(t) \rangle$ can be written, when $\omega = 0$, as the sum of four terms

$$\langle C^{2}(p) \rangle \tau_p + \langle C(p) D'(q) \rangle \tau_q + \langle C'(p) D(q) \rangle \tau_p + \langle D(q) D'(q) \rangle \tau_q$$

The factors accompanying these terms refer not to a single instant of time, while the quantities with and without a prime, have different summation indices.

Averaging we find, that the second and third term both vanish. No difficulty is experienced in averaging the first term.

We shall use the following identity [7] to simplify the term $\langle D(q) D'(q) \rangle$:

$$\langle \mathbf{K}_\nu A \rangle = - kT \langle \partial A / \partial q^\nu \rangle$$

where A denotes any function of the coordinates and $\mathbf{K}_\nu$ is the force exerted on the νth molecule by all the remaining molecules. Double index distribution functions are then used (method of conditional distributions [8 and 9]) to perform the statistical averaging, with the following final result:

$$\eta = \frac{kT\tau_p}{2v} + \tau_q \frac{2\pi}{15v^2} \int_{r_0}^{\infty} \frac{d}{dr}\left(r^4 \frac{d\Phi}{dr}\right) \varphi(r)\, dr \qquad \left(\frac{4\pi}{3} r_0^3 = v\right) \tag{17}$$

where v is the molecular volume, r_0 is the radius of the molecular cell and $\varphi(r)$ is a statistical distribution function to which we shall return later.

We note that the integral appearing in (17) defines the shear modulus of the fluid subjected to a cyclic loading at the upper frequency limit [10].

Thus, in place of a single problem of viscosity determination, we have two separate problems, one of which is concerned with the necessity of carrying out the statistical averaging process when the system is in equilibrium, and the other with the determination of the relaxation time of the system.

A statistical method for determining the relaxation time τ_q (for simple fluids it has the order of 10^{-12} sec.) was given in [8].

The values of τ_q agree with the thermal estimates [9], therefore we shall limit ourselves to the determination of τ_p.

Following the general scheme of Kirkwood [11] but using the double index distribution functions mentioned above, we can repeat the statistical derivation of the Langevin equation

$$d\mathbf{p}/dt = - \xi / m\, \mathbf{p} + \mathbf{G}(t) \tag{18}$$

where $\mathbf{p}$ is the impulse of a particle of mass m, and $\mathbf{G}(t)$ is a random force acting on a particle and averaging over the Gibbs distribution to zero. The coefficient of friction ξ of a particle is equal to the following integral of the autocorrelation function:

$$\xi = \frac{1}{3kT} \int_0^{\infty} \langle \mathbf{K}(0)\, \mathbf{K}(s) \rangle\, ds \tag{19}$$

where $\mathbf{K}(s)$ is the force exerted on the given particle at the instant s by all the remaining particles of the system. Expression (19) can be approximately represented by

$$\xi = \frac{1}{3kT} \langle K^2 \rangle \tau_q \tag{20}$$

or in accordance with [9], by

$$\xi = \frac{4\pi\tau_q}{3kTv} \int_{r_0}^{\infty} [\Phi'(r)]^2 \varphi(r)\, r^2\, dr, \qquad \varphi(r) = v\, \frac{F_{11}^{(1)}(\mathbf{q}^1, \mathbf{q}^2)}{F_{11}(\mathbf{q}^1)} \tag{21}$$

We recall that, in the statistical scheme used by us, $F_{11}^{(1)}$ denotes the probability density that two molecular volumes v_1 and v_2 surrounding, respectively, the coordinates $\mathbf{q}^1 \subset v_1$ and $\mathbf{q}^2 \subset v_2$ contain any two molecules, the remaining cells also containing a single molecule each (we call this approximation F_{11}). Determining similarly the probability density for a single particle $F_{11}(\mathbf{q}^1)$ we see, that $v^{-1}\varphi(r)$ plays the part of the conditional distribution function $F_{11}(\mathbf{q}^2 / \mathbf{q}^1)$. In a number of specific applications, the fixed point is chosen at the center of a sphere of volume v_1.

Using (18) we obtain the characteristic impulse relaxation time

$$\tau_p = m/\xi \tag{22}$$

There is a difference of about two orders of magnitude between τ_p and τ_q. For example, for Hydrogen we have $\tau_p = 3.8\times10^{-14}$ sec. at $T = 14.89°K$ and $\tau_p = 5.39\times10^{-14}$ sec. at $T = 19.92°K$, while for Argon we have $\tau_p = 4.798\times10^{-14}$ sec. at $T = 90.03$ °K.

Combining (20) and (22) we obtain an inverse relationship between τ_p and τ_q, given by

$$\tau_p = \frac{mkT}{\langle K^2\rangle \tau_q} \qquad (23)$$

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ON THE FORMULATION OF THE DEFINING EQUATIONS FOR A PLANE FLOW OF A CONTINUOUS MEDIUM WITH DRY FRICTION

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Construction of the defining equations for the critical equilibrium state of an incompressible, continuous, free-running medium with dry friction [1], presents certain fundamental difficulties. As we know, the lines (areas) of the critical equilibrium state (along the tangents up to which the Coulomb's condition holds), are situated symmetrically, at an angle α to the direction of the highest normal stress and only when the angle of internal friction is equal to zero $\Phi = 0$ (the case of the perfectly plastic body), then the lines of the